Homework 9 Solutions

Math 131B-1

- We see that $||f+g||_2^2 = \langle f+g, f+g\rangle = \langle f,f\rangle + \langle f,g\rangle + \langle g,f\rangle + \langle g,g\rangle = \langle f,f\rangle + \langle g,g\rangle = ||f||_2^2 + ||g||_2^2$.

- We know $|g(x)| < M$ for some $M > 0$. Therefore

$$\begin{aligned}|f*g(x) - f*g(x')| &= \left|\int_0^1 f(y)g(x-y)dy - \int_0^1 f(y')g(x'-y)dy\right| \\ &\leq 2M\left|\int_0^1 (g(x-y) - g(x'-y))dx\right|.\end{aligned}$$

  Now, since g is uniformly continuous, given $\epsilon > 0$ there is some δ such that if $|x-x'| < \delta$, $|g(x-y) - g(x'-y)| < \frac{\epsilon}{2M}$. So as long as $|x-x'| < \delta$, we have $|f*g(x) - f*g(x')| < \epsilon$.

- (16.2.3) Let $f \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ be nonzero. Let $a = ||f||_\infty = \sup_{x\in[0,1)}|f(x)|$. Then in particular $|f(x)| \leq a$ on $[0,1]$, so $f(x)\overline{f(x)} = |f(x)|^2 \leq a^2$ on $[0,1]$. Therefore we see that $||f||_2^2 = \int_0^1 f(x)\overline{f(x)}dx \leq a^2(1-0) = ||f||_\infty^2$. We conclude that $||f||_2 \leq ||f||_\infty$.

  Now let A, B be real numbers such that $0 < A \leq B$. Let $g(x)$ be a continuous 1-periodic function with $||g||_\infty = k$, $\int_0^1 g = \ell$, and $kA^2 - \ell B^2 > 0$. (We can always find such a g, for example by squaring the functions in Problem 16.2.6(d) below.) Now let $h(x) = \sqrt{c+dg}$, where $c = \frac{kA^2-\ell B^2}{k-\ell}$, and $d = \frac{B^2-A^2}{k-\ell}$. Note that both c and d are positive. These numbers are chosen so that $||h||_\infty = \sup_{x\in[0,1]} h = \sqrt{c+kd} = B$ and $||h||_2 = \sqrt{\int_0^1 c+dg} = \sqrt{c+\ell d} = \sqrt{A^2} = A$.

- (16.2.6) (a) Because $0 \leq ||f_n - f||_2 \leq ||f_n - f||_\infty$ and uniform convergence is convergence in the L^∞ metric, uniform convergence implies L^2 convergence.

  (b) Let $f_n(x) = \sqrt{2nx}$ for $x \in [0, \frac{1}{2n}]$, $f(x) = \sqrt{2-2nx}$ for $x \in [\frac{1}{2n}, \frac{1}{n}]$, and $f_n(x) = 0$ for the remainder of the interval $[0,1]$, and extend periodically. Then $f_n \to 0$ pointwise, but not uniformly (because $f_n(\frac{1}{2n}) = 1$ for all n). Moreover, $\sqrt{\int_0^1 f^2(x)dx} = \sqrt{\frac{1}{2n}} \to 0$, so $f_n \to f$ in the L^2 metric.

  (c)Let $f_n(x) = (2x)^n$ on $[0,\frac{1}{2}]$ and $f_n(x) = (2-2x)^n$ on $[\frac{1}{2},1]$, and extend periodically. Then $f_n(\frac{1}{2}) = 1$ for all n, so f_n does not converge to the zero function pointwise, but $\int_0^1 f_n(x)^2dx = \frac{1}{2n+1}$, so in the L^2 metric, the functions f_n converge to 0.

(d) Let $f_n(x) = \sqrt{2n^2 x}$ for $x \in [0, \frac{1}{2n}]$, $f_n(x) = \sqrt{2 - 2n^2 x}$ for $x \in [\frac{1}{2n}, \frac{1}{n}]$, and $f_n(x) = 0$ for the remainder of $[0,1]$, and extend periodically. Then $f_n \to 0$ pointwise, but $\int_0^1 f_n^2 = \frac{1}{2}$ for all n, so f_n does not converge to 0 in the L^2 metric.

- (Tao 16.5.1) Observe that

$$\begin{aligned}\sum_{n=\infty}^{\infty} \hat{f}(n) e_n &= \sum_{n=-\infty}^{\infty} \hat{f}(n)(\cos(2\pi nx) + i\sin(2\pi nx)) \\ &= \hat{f}(0) + \sum_{n=1}^{\infty}[(\hat{f}(n) + \hat{f}(-n))\cos(2\pi nx) + (\hat{f}(n) - \hat{f}(-n)) i \sin(2\pi nx)]\end{aligned}$$

Here we have used the fact that $\cos x$ is even and $\sin x$ is odd. However, $\hat{f}(0) = \int_0^1 f(x) e^{2\pi i(0)x} dx = \int_0^1 f(x)(1) dx = \int_0^1 f(x)\cos(2\pi(0)x) dx = \frac{a_0}{2}$, and for the general case, we have

$$\begin{aligned}\hat{f}(n) + \hat{f}(-n) &= \int_0^1 f(x) e^{-2\pi i nx} dx + \int_0^1 f(x) e^{2\pi nx} dx \\ &= \int_0^1 f(x)[\cos(2\pi nx) - i\sin(2\pi nx)] dx + \int_0^1 f(x)[\cos(2\pi nx) + i\sin(2\pi nx)] dx \\ &= 2\int_0^1 f(x)\cos(2\pi nx) dx \\ &= a_n\end{aligned}$$

Similarly, $\hat{f}(n) - \hat{f}(-n) = -2ib_n$. Therefore $\sum_{n=\infty}^{\infty} \hat{f} e^{2\pi i nx} = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(2\pi nx) + b_n \sin(2\pi nx))$, and converges in the L^2 metric to f.

(b) If $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are absolutely convergent, then

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)| = \sum_{n=-\infty}^{-1} \frac{1}{2}|a_n + ib_n| + \sum_{n=0}^{\infty} \frac{1}{2}|a_n - ib_n|$$

is also absolutely convergent. Therefore we may apply Theorem 16.5.3.

- (Tao 16.5.2) Let $f(x) = (1-2x)^2$ on $[0,1)$ and $\mathbb{Z}$-periodic.

(a) We use integration by parts to compute a_n and b_n, and get the series shown. Since $\sum b_n = \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2}$ converges absolutely, by Exercise 16.5.1, we get uniform convergence of the Fourier series.

(b) At $x = 0$ we have $1 = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2}$. Solving gives the expected result.

(c) Because $\cos(2\pi inx) = \frac{e_n + e_{-n}}{2}$, the Fourier coefficients of $f(x) = (1-2x)^2$ are $\hat{f}(0) = \frac{1}{3}$ and $\hat{f}(n) = \hat{f}(-n) = \frac{2}{\pi^2 n^2}$. By the Plancherel Theorem, the series $\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2$ is absolutely convergent. Because absolutely convergent series can be rearranged, we can rewrite this series as $\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 = \frac{1}{9} + \sum_{n=1}^{\infty} \frac{8}{\pi^4 n^4}$. Moreover, we know this series converges to $||f||_2^2 = \frac{1}{5}$. Ergo, $\frac{1}{5} = \frac{1}{9} + \sum_{n=1}^{\infty} \frac{8}{\pi^4 n^4}$. Solving gives the expected result.

- (Tao 16.5.4) The interesting point is computing $\widehat{f'}$. We know that $\widehat{f'}(n) = \int_0^1 f'(x)e^{-2\pi inx}dx$. After integration by parts, this integral becomes

$$\widehat{f'}(n) = f(x)e^{-2\pi inx}|_0^1 + \int_0^1 (2\pi in) f(x) e^{-2\pi inx} dx = 0 + 2\pi in\hat{f}(x)$$

as promised.